

# Exact sequences in the cohomology of a group extension

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Dedicated to the memory of my friend and colleague Chi Han Sah

## Abstract

In [J. of Alg. 369: 70-95, 2012], the authors constructed a seven term exact sequence in the cohomology of a group extension. We show that the maps in this seven term exact sequence other than the obvious inflation and restriction maps are special cases of maps constructed in two papers published more than 30 years ago and that the tools developed in these papers also settle the issues left open in [op. cit.]; in particular, these tools establish the precise link between the maps in [op. cit.] and the differentials in the spectral sequence.

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# 1 Introduction

Consider an extension of (discrete) groups

$$e_G: 1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1 \quad (1.1)$$

and a  $G$ -module  $M$ . In [Sah74] ((3) p. 257), C. H. Sah extended the classical five term exact sequence in the cohomology of the group extension (1.1) with coefficients in  $M$  by two more terms to an exact sequence of the kind

$$\begin{aligned} 0 &\longrightarrow H^1(Q, M^N) \xrightarrow{\inf} H^1(G, M) \xrightarrow{\text{res}} H^1(N, M)^Q \\ &\xrightarrow{d_2} H^2(Q, M^N) \xrightarrow{\inf} H^2(G, M)_1 \xrightarrow{\text{tg}} H^1(Q, H^1(N, M)) \\ &\xrightarrow{d_2} H^3(Q, M^N). \end{aligned} \quad (1.2)$$

Here, as usual, for  $K = N$  as well as for  $K = Q$ , the notation  $-^K$  refers to the invariants relative to the group  $K$ , the notation  $\text{res}$  and  $\inf$  refer to the maps in cohomology induced by the injection of  $N$  into  $G$  and by the projection from  $G$  to  $Q$ , respectively, and  $H^2(G, M)_1$  denotes the kernel of  $\text{res}: H^2(G, M) \rightarrow H^2(N, M)^Q$ .

Let

$$\lambda: H^1(Q, H^1(N, M)) \longrightarrow H^3(Q, M^N) \quad (1.3)$$

be the map constructed in [Hue81a] that yields an explicit description of the differential

$$d_2: H^1(Q, H^1(N, M)) \longrightarrow H^3(Q, M^N). \quad (1.4)$$

In [DHW12], the authors constructed explicit maps

$$\text{tr}: H^1(N, M)^Q \longrightarrow H^2(Q, M^N) \quad (1.5)$$

$$\rho: H^2(G, M)_1 \longrightarrow H^1(Q, H^1(N, M)) \quad (1.6)$$

such that, with  $\text{tr}$ ,  $\rho$ , and  $\lambda$  substituted for, respectively,

$$d_2: H^1(N, M)^Q \longrightarrow H^2(Q, M^N), \quad (1.7)$$

$$\text{tg}: H^2(G, M)_1 \longrightarrow H^1(Q, H^1(N, M)), \quad (1.8)$$

$$d_2: H^1(Q, H^1(N, M)) \longrightarrow H^3(Q, M^N), \quad (1.9)$$

an exact sequence of the kind (1.2) results. On p. 71 (line 3 after the diagram), they claim “The inflation and the restriction maps in the sequence are well understood, but the others are induced by differentials in the spectral sequence and there is no explicit description available, except from Huebschmann’s description of  $\lambda$ ”. Thereafter (p. 71, 4th paragraph), they note “In general we do not know whether or not the maps in our sequence coincide with the ones from the spectral sequence.” In the addendum [DHW13] to [DHW12], they point out the following: “Both our paper and [Hue81b] (Section 1.5) give a conceptual description of a map

$$\text{tr}: H^1(N, M)^Q \longrightarrow H^2(Q, M^N) \quad (1.10)$$

appearing in exact sequences. However the descriptions obtained are actually quite different.” The paper [Hue81b] is not quoted in the original paper [DHW12]; it is quoted (with a precise reference) in [Hue81a], however.

The purpose of this note is to show that the maps (1.5) and (1.6) are special cases of maps given in [Hue81a] and [Hue81b], see Subsection 2.4 below, that the requisite explicit descriptions may be found in [Hue81a] and [Hue81b], and that all the issues addressed and left open in [DHW12] and [DHW13] have been settled already in [Hue81a] and [Hue81b]. In particular, Proposition 3.1 in [Hue81a] entails an explicit conceptual description of the differential (1.7), and another conceptual description of this differential is given in [Hue81b] (Theorem 6, Section 5). Moreover, in Subsection 2.5 below we will show that the description of (1.10) in [DHW12] is essentially the same as that in [Hue81b]. For intelligibility we note that the main result in [Hue81b] is an extension of the classical five term exact sequence by three more terms, viz.

$$\begin{array}{ccccccc} & & & \longrightarrow & H^1(N, M)^Q & & \\ \Delta \longrightarrow & H^2(Q, M^N) & \xrightarrow{\inf} & H^2(G, M) & \xrightarrow{j} & \text{Xpext}(G, N; M) & (1.11) \\ \Delta \longrightarrow & H^3(Q, M^N) & \xrightarrow{\inf} & H^3(G, M). & & & \end{array}$$

The description of (1.6) given in [DHW12] can be deduced from the construction of the map  $j$  by inspection, see Section 3 below. The conceptual descriptions of the differentials in [Hue81a] and the construction of the exact sequence (1.11) rely on the interpretation of group cohomology in terms of crossed modules developed in [Hue80].

We dedicate this paper to the memory of Chi Han Sah, see e.g., [DPR<sup>+</sup>98]. He contributed substantially to the understanding of the spectral sequence of a group extension, cf. [Sah74], [Sah77].

## 2 $d_2: H^1(N, M)^Q \longrightarrow H^2(Q, M^N)$

We adjust the notation to the present discussion. We denote the group ring of  $G$  by  $\mathbb{Z}G$ , the standard augmentation map by  $\varepsilon: \mathbb{Z}G \rightarrow \mathbb{Z}$ , and the augmentation ideal  $\ker(\varepsilon)$  by  $IG$ , and we use the same kind of notation for the group ring of  $Q$  and the augmentation ideal of  $Q$ .

### 2.1 Description in [Hue81a]

Let  $T = \ker(\text{Hom}_N(IG, M) \rightarrow \text{Ext}_N^1(\mathbb{Z}, M) \cong H^1(N, M))$ ; with respect to the obvious  $Q$ -module structures on  $\text{Hom}_N(IG, M)$  and  $\text{Ext}_N^1(\mathbb{Z}, M)$ , the map from  $\text{Hom}_N(IG, M)$  to  $\text{Ext}_N^1(\mathbb{Z}, M)$  is a morphism of  $Q$ -modules, and hence  $T$  inherits a  $Q$ -module structure.

Let

$$e_M: 0 \longrightarrow M \xrightarrow{i_M} E_M \xrightarrow{p_M} \mathbb{Z} \longrightarrow 0 \quad (2.1)$$

represent a member of  $\text{Ext}_N(\mathbb{Z}, M)^Q \cong H^1(N, M)^Q$ . Lift the identity of  $\mathbb{Z}$  to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & IG & \longrightarrow & \mathbb{Z}G & \xrightarrow{\varepsilon} & \mathbb{Z} \longrightarrow 0 \\ & & \mu \downarrow & & \nu \downarrow & & \parallel \\ 0 & \longrightarrow & M & \xrightarrow{i_M} & E_M & \xrightarrow{p_M} & \mathbb{Z} \longrightarrow 0 \end{array} \quad (2.2)$$

in the category of  $N$ -modules. For  $x \in G$ , let  $\alpha_x$  denote the associated operator on  $IG$  as well as that on  $M$  where the notation is slightly abused; given  $q \in Q$ , pick  $x \in G$  such that  $\pi(x) = q$ , and define the derivation

$$d: Q \longrightarrow \text{Hom}_N(IG, M), \quad d(q) = \alpha_x \mu \alpha_x^{-1} - \mu, \quad q \in Q. \quad (2.3)$$

Each value  $d(q)$ , as  $q$  ranges over  $Q$ , is well defined, that is, does not depend on the choice of  $x \in G$  such that  $\pi(x) = q$ , and the values of  $d$  lie in  $T$  since  $[e_M] \in \text{Ext}_N(\mathbb{Z}, M)$  is  $Q$ -invariant. Consider the semi-direct fiber product group

$$\text{Hom}_N(\mathbb{Z}G, M) \rtimes_T Q = \{(\varphi, q); \varphi|_G = d(q)\} \subseteq \text{Hom}_N(\mathbb{Z}G, M) \rtimes Q. \quad (2.4)$$

The projection to  $Q$  yields the group extension

$$\widehat{e}_M: 0 \longrightarrow M^N \longrightarrow \text{Hom}_N(\mathbb{Z}G, M) \rtimes_T Q \longrightarrow Q \longrightarrow 1. \quad (2.5)$$

In view of Proposition 3.1 in [Hue81a], the assignment to  $e_M$  of the class  $[\widehat{e}_M] \in H^2(Q, M^N)$  represented by  $\widehat{e}_M$  yields a conceptual description of the differential (1.7).

*Remark 2.1.* A cocycle description of the differential (1.7) is given in Remark 3.2 of [Hue81a]. For the special case where  $N$  acts trivially on  $M$ , a similar cocycle description can be found on p. 21 of [Sah77]. In the latter paper, C. H. Sah explored some of the differentials in the spectral sequence of a split group extension via certain characteristic classes. These characteristic classes have been extended in [Hue89]; in that paper, the interested reader will also find more references related with these characteristic classes.

## 2.2 Description in [Hue81b]

We start quoting a phrase from [DHW13]: “Indeed, in [Hue81b] (Section 1.5)  $H^1(N, M)$  is interpreted as being  $\text{Ext}_{\mathbb{Z}N}(\mathbb{Z}, M)$  and so consists of equivalence classes of extensions of modules. In contrast, the construction of our map is much more elementary.”

We now recall the construction in [Hue81b] (Section 5). Let  $\overline{\text{Aut}(e_M)}$  denote the group of pairs  $(\alpha, x)$  such that  $\alpha$  is an automorphism of  $E_M$  whose restriction  $\alpha_M$  to  $M$  is an automorphism of  $M$  and coincides with the automorphism  $x_M$  of  $M$  induced by  $x \in G$  via the  $G$ -action on  $M$ . Since the class  $[e_M] \in \text{Ext}_N(\mathbb{Z}, M)$  is  $Q$ -invariant, every  $x \in G$  lifts to a member  $(\alpha, x)$  of  $\overline{\text{Aut}(e_M)}$ . Moreover, given  $m \in M$ , let  $\alpha_m: E_M \rightarrow E_M$  be the automorphism of  $E_M$  given by

$$\alpha_m(y) = y + p_M(y)m, \quad y \in E_M. \quad (2.6)$$

The assignment to  $m \in M$  of  $(\alpha_m, e) \in \overline{\text{Aut}(e_M)}$  and to  $(\alpha, x) \in \overline{\text{Aut}(e_M)}$  of  $x \in G$  yields a group extension

$$0 \longrightarrow M \xrightarrow{i_M} \overline{\text{Aut}(e_M)} \xrightarrow{p_M} G \longrightarrow 1. \quad (2.7)$$

The extension  $e_M$  being one of  $N$ -modules, the  $N$ -action  $n \mapsto \alpha_n: E_M \rightarrow E_M$  ( $n \in N$ ) on  $E_M$  yields the injection

$$s: N \longrightarrow \overline{\text{Aut}(e_M)}, \quad n \longmapsto (\alpha_n, n), \quad (2.8)$$

and the composite of this injection with the projection to  $G$  coincides with the injection  $N \rightarrow G$  in (1.1). By construction, the normaliser  $\text{Aut}(e_M)$  of  $s(N)$  in  $\overline{\text{Aut}(e_M)}$  consists of the pairs  $(\alpha, x) \in \overline{\text{Aut}(e_M)}$  such that

$$\alpha \alpha_n \alpha^{-1} = \alpha_{x n x^{-1}}, \quad n \in N.$$

By [Hue81b] (Proposition 1.7),  $\text{Aut}(e_M)$  still maps onto  $G$ , and the surjection onto  $G$  yields a group extension

$$0 \longrightarrow M^N \longrightarrow \text{Aut}(e_M) \xrightarrow{\pi_{e_M}} G \longrightarrow 1. \quad (2.9)$$

Passing to quotients, we obtain the group extension

$$\tilde{e}_M: 0 \longrightarrow M^N \longrightarrow \text{Out}(e_M) \longrightarrow Q \longrightarrow 1; \quad (2.10)$$

here  $\text{Out}(e_M) = \text{Aut}(e_M)/s(N)$ . By Theorem 6 in [Hue81b], the assignment to  $e_M$  of  $\tilde{e}_M$  yields a homomorphism

$$\Delta: H^1(N, M)^Q \longrightarrow \text{Opext}(Q, M^N) \quad (2.11)$$

and, via the canonical isomorphism  $\text{Opext}(Q, M^N) \cong H^2(Q, M^N)$ , the homomorphism  $\Delta$  yields a conceptual description of the differential  $d_2: H^1(N, M)^Q \longrightarrow H^2(Q, M^N)$ .

### 2.3 Relationship between the two previous descriptions

Pulling back the group (2.4) through the projection  $G \rightarrow Q$  in (1.1), we obtain the group

$$\text{Hom}_N(\mathbb{Z}G, M) \rtimes_T G = \{(\varphi, x); \varphi|_{IG} = \alpha_x \mu \alpha_x^{-1} - \mu\} \subseteq \text{Hom}_N(\mathbb{Z}G, M) \rtimes G. \quad (2.12)$$

Given  $(\varphi, y) \in \text{Hom}_N(\mathbb{Z}G, M) \rtimes G$ , define

$$\begin{aligned} \alpha_1: \mathbb{Z}G &\rightarrow E_M, \quad \alpha_1(x) = \varphi(yx) + \nu(yx), \quad x \in \mathbb{Z}G, \\ \alpha_2: E_M &\rightarrow E_M, \quad \alpha_2(m) = ym. \end{aligned}$$

Since the left-hand square of (2.2) is a pushout diagram of abelian groups,  $\alpha_1$  and  $\alpha_2$  induce a unique automorphism

$$\alpha_{\varphi, y}: E_M \longrightarrow E_M.$$

By Lemma 5.1 of [Hue81b], the assignment to  $(\varphi, y) \in \text{Hom}_N(\mathbb{Z}G, M) \rtimes_T G$  of  $\alpha_{\varphi, y}$  yields a morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^N & \longrightarrow & \text{Hom}_N(\mathbb{Z}G, M) \rtimes_T G & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \Phi \downarrow & & \parallel \\ 0 & \longrightarrow & M^N & \longrightarrow & \text{Aut}(e_M) & \longrightarrow & G \longrightarrow 1 \end{array} \quad (2.13)$$

of group extensions. The resulting morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^N & \longrightarrow & \text{Hom}_N(\mathbb{Z}G, M) \rtimes_T Q & \longrightarrow & Q \longrightarrow 1 \\ & & \parallel & & \hat{\Phi} \downarrow & & \parallel \\ 0 & \longrightarrow & M^N & \longrightarrow & \text{Out}(e_M) & \longrightarrow & Q \longrightarrow 1 \end{array} \quad (2.14)$$

of group extensions identifies  $\text{Out}(e_M)$  with  $\text{Hom}_N(\mathbb{Z}G, M) \rtimes_T Q$ . In view of Proposition 3.1 in [Hue81a], diagram (2.14) entails a proof of Theorem 6 in [Hue81b].

### 2.4 Construction in [DHW12]

The construction of (1.5) in Section 5 comes down to this: Consider the split extension

$$0 \longrightarrow M \longrightarrow M \rtimes G \longrightarrow G \longrightarrow 1. \quad (2.15)$$

A section  $s: N \rightarrow M \rtimes G$  whose composite with the projection to  $G$  coincides with the injection  $N \rightarrow G$  in  $e_G$  is determined by its component  $\varphi: N \rightarrow M$ , and  $\varphi$  is a 1-cocycle on

$N$  with values in  $M$  that represents a  $Q$ -invariant cohomology class, i. e.,  $[\varphi] \in H^1(N, M)^Q$ . Given such a section  $s$  or, equivalently, 1-cocycle  $\varphi$ , the normalizer  $N_{M \rtimes G}(sN)$  of  $sN$  in  $M \rtimes G$  maps onto  $G$ ,

$$0 \longrightarrow M^N \longrightarrow N_{M \rtimes G}(sN) \longrightarrow G \longrightarrow 1 \quad (2.16)$$

is exact, and so is

$$\tilde{e}: 0 \longrightarrow M^N \longrightarrow N_{M \rtimes G}(sN)/sN \longrightarrow Q \longrightarrow 1. \quad (2.17)$$

In [DHW12] the map (1.5) is essentially defined by the assignment to an  $M$ -valued 1-cocycle  $\varphi$  on  $N$  representing a  $Q$ -invariant cohomology class in  $H^1(N, M)$  of the class of the extension  $\tilde{e}$ . The construction in [DHW12] is actually couched in what is referred to as “partial semi-direct complements”. The 1-cocycle  $\varphi$  is lurking behind the choice of the subgroup denoted by  $H$ , indeed, it is the component into  $M$  of the resulting section  $s: N \rightarrow M \rtimes G$  such that  $H = sN$ .

## 2.5 Comparison of the constructions in [Hue81b] and [DHW12]

We maintain the notation established before. Thus  $e_M$  is an extension of the kind (2.1) representing a class  $[e_M] \in \text{Ext}_N^1(\mathbb{Z}, M)^Q \cong H^1(N, M)^Q$ , and  $s: N \rightarrow M \rtimes G$  is a section as in Subsection 2.4.

As an abelian group,  $E_M$  decomposes as a direct sum  $M \oplus \mathbb{Z}$ . Such a decomposition induces an obvious  $G$ -module structure on  $E_M$  that restricts to the  $G$ -module structure on  $M$ . This  $G$ -module structure induces a section  $\sigma: G \rightarrow \overline{\text{Aut}}(e_M)$  for the extension (2.7) and hence an isomorphism  $\overline{\text{Aut}}(e_M) \rightarrow M \rtimes G$ . This isomorphism induces a morphism

$$\begin{array}{ccccccc} \tilde{e}: 0 & \longrightarrow & M^N & \longrightarrow & N_{M \rtimes G}(sN)/sN & \longrightarrow & Q \longrightarrow 1 \\ & & \parallel & & \downarrow & & \parallel \\ \tilde{e}_M: 0 & \longrightarrow & M^N & \longrightarrow & \text{Out}(e_M) & \longrightarrow & Q \longrightarrow 1 \end{array} \quad (2.18)$$

of group extensions. Hence, cf. the introductory phrase of Subsection 2.2 above, the construction in Section 5 of [DHW12] that underlies the map (1.5) is exactly the same as that in [Hue81b] which underlies the map (2.11). The only difference is that the interpretation of the naive semi-direct product  $M \rtimes G$  as the group  $\overline{\text{Aut}}(e_M)$ , not noticed in [DHW12], enabled us to show in [Hue81b] that (2.11) actually yields the differential (1.7). In particular, the map (1.5) coincides with the differential (1.7), issue left open in [DHW12].

## 3 $H^2(G, M)_1 \longrightarrow H^1(Q, H^1(N, M))$

We begin by repeating a quote from the introduction of [DHW12]: “In general we do not know whether or not the maps in our sequence coincide with the ones from the spectral sequence.”

The construction of (1.6) in [DHW12] proceeds as follows: Consider a group extension

$$e: 0 \longrightarrow M \longrightarrow E \xrightarrow{p} G \longrightarrow 1 \quad (3.1)$$

whose restriction to  $N$  splits, and let  $s: N \rightarrow E$  be a homomorphism such that  $p \circ s$  is the identity of  $N$ . The assignment to  $x \in E$  of

$$d_x: N \longrightarrow M, \quad d_x(n) = xs(x^{-1}nx)s(n^{-1}),$$

is a derivation  $E \rightarrow \text{Der}(N, M)$  which passes to a derivation  $\delta_e: Q \rightarrow H^1(N, M)$  and, in [DHW12] (Section 6), (1.6) is defined by the assignment to  $e$  of  $\delta_e$ .

We do not reproduce here the construction of the group  $\text{Xpext}(G, N; M)$  in Section 2 of [Hue81b] nor that of the homomorphism  $\alpha: H^1(Q, H^1(N, M)) \rightarrow \text{Xpext}(G, N; M)$  in Subsection 1.3 of [Hue81b] and of the homomorphisms  $j$  and  $\Delta$  below. The various groups under discussion fit into the commutative diagram

$$\begin{array}{ccccc} H^2(G, M)_1 & \xrightarrow{\rho} & H^1(Q, H^1(N, M)) & \xrightarrow{d_2} & H^3(Q, M^N) \\ \downarrow & & \alpha \downarrow & & \parallel \\ H^2(G, M) & \xrightarrow{j} & \text{Xpext}(G, N; M) & \xrightarrow{\Delta} & H^3(Q, M^N) \end{array} \quad (3.2)$$

with exact rows, cf. diagram (1.10) in [Hue81b]. The homomorphism  $\alpha$  is injective whence  $\rho$  is a special case of  $j$ , and the exact sequence in [DHW12] can be deduced from the exact sequence (1.11) constructed in [Hue81b]. The commutativity of (3.2) entails that  $\rho$  maps onto

$$E_{\infty}^{1,1} = \ker(d_2: H^1(Q, H^1(N, M)) \rightarrow H^3(Q, M^N))$$

whence  $\rho$  “coincides with the map from the spectral sequence”, issue left open in [DHW12].

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